CONSTRUCTIONS OF POTENTIALLY EVENTUALLY POSITIVE SIGN PATTERNS WITH REDUCIBLE POSITIVE PART

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Abstract. Potentially eventually positive (PEP) sign patterns were introduced in “Sign patterns that allow eventual positivity,” Electronic Journal of Linear Algebra, 19 (2010): 108–120, where it was noted that $A$ is PEP if its positive part is primitive, and an example was given of a $3 \times 3$ PEP sign pattern with reducible positive part. We extend these results by constructing $n \times n$ PEP sign patterns with reducible positive part, for every $n \geq 3$.

Key words. potentially eventually positive, PEP, sign pattern, matrix, digraph


1. Introduction. A sign pattern matrix (or sign pattern) is a matrix having entries in $\{+, -, 0\}$. For a real matrix $A$, $\text{sgn}(A)$ is the sign pattern having entries that correspond to the signs of the entries in $A$. If $A$ is an $n \times n$ sign pattern, the qualitative class of $A$, denoted $Q(A)$, is the set of all $A \in \mathbb{R}^{n \times n}$ such that $\text{sgn}(A) = A$, where $\text{sgn}(A) = [\text{sgn}(a_{ij})]$; such a matrix $A$ is called a realization of $A$. Qualitative matrix problems were introduced more than sixty years ago by Samuelson in the mathematical modeling of problems from economics [7]. Sign pattern matrices have useful applications in economics, population biology, chemistry and sociology. If $P$ is a property of a real matrix, then a sign pattern $A$ is potentially $P$ (or allows $P$) if there is some $A \in Q(A)$ that has property $P$.

The spectrum of a square matrix $A$, denoted $\sigma(A)$, is the multiset of the eigenvalues of $A$, and the spectral radius of $A$ is defined as $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$.

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Matrix $A$ has the strong Perron-Frobenius property if $\rho(A) > 0$ is a simple strictly dominant eigenvalue of $A$ that has a positive eigenvector. A matrix $A \in \mathbb{R}^{n \times n}$ is eventually positive if there exists a $k_0 \in \mathbb{Z}^+$ such that for all $k \geq k_0$, $A^k > 0$, where the inequality is entrywise. Handelman developed the following test for eventual positivity in [2]: a matrix $A$ is eventually positive if and only if both $A$ and $A^T$ satisfy the strong Perron-Frobenius property. If there exists a $k$ such that $A^k > 0$ and $A^{k+1} > 0$, then $A$ is eventually positive [4]. A sign pattern $\mathcal{A}$ is potentially eventually positive (PEP) if there exists an eventually positive realization $A \in Q(\mathcal{A})$.

For a sign pattern $\mathcal{A} = [\alpha_{ij}]$, define the positive part of $\mathcal{A}$ to be $\mathcal{A}^+ = [\alpha^+_{ij}]$ and the negative part of $\mathcal{A}$ to be $\mathcal{A}^- = [\alpha^-_{ij}]$, where

$$
\alpha^+_{ij} = \begin{cases} + & \text{if } \alpha_{ij} = +, \\
0 & \text{if } \alpha_{ij} = 0 \text{ or } \alpha_{ij} = -. \end{cases} \quad \text{and} \quad \alpha^-_{ij} = \begin{cases} - & \text{if } \alpha_{ij} = -, \\
0 & \text{if } \alpha_{ij} = 0 \text{ or } \alpha_{ij} = +. \end{cases}
$$

Clearly $\mathcal{A} = \mathcal{A}^+ + \mathcal{A}^-$. For a matrix $A \in \mathbb{R}^{n \times n}$, the positive part $A^+$ of $A$ and negative part $A^-$ of $A$ are defined analogously, and $A = A^+ + A^-$. 

A digraph $\Gamma = (V, E)$ consists of a finite, nonempty set $V$ of vertices, together with a set $E \subseteq V \times V$ of arcs. Note that a digraph allows loops (arcs of the form $(v, v)$) and may have both arcs $(v, w)$ and $(w, v)$ but not multiple copies of the same arc. Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$. The digraph of $A$, denoted $\Gamma(A)$, has vertex set $\{1, \ldots, n\}$ and arc set $\{(i, j) : a_{ij} \neq 0\}$. If $\mathcal{A}$ is a sign pattern, then $\Gamma(\mathcal{A}) = \Gamma(A)$ where $A \in Q(\mathcal{A})$. A digraph $\Gamma$ is strongly connected if for any two distinct vertices $v$ and $w$ of $\Gamma$, there is a path in $\Gamma$ from $v$ to $w$.

A square matrix $A$ is reducible if there exists a permutation matrix $P$ such that

$$
PAP^T = \begin{bmatrix} A_{11} & 0 \\
A_{21} & A_{22} \end{bmatrix}
$$

where $A_{11}$ and $A_{22}$ are nonempty square matrices and $0$ is a (possibly rectangular) block consisting entirely of zero entries, or $A$ is the $1 \times 1$ zero matrix. If $A$ is not reducible, then $A$ is called irreducible. It is well known that for $n \geq 2$, $A$ is irreducible if and only if $\Gamma(A)$ is strongly connected. For a strongly connected digraph $\Gamma$, the index of imprimitivity is the greatest common divisor of the lengths of the cycles in $\Gamma$. A strongly connected digraph is primitive if its index of imprimitivity is one; otherwise it is imprimitive. The index of imprimitivity of a nonnegative sign pattern $\mathcal{A}$ is the index of imprimitivity of $\Gamma(\mathcal{A})$ and $\mathcal{A} \geq 0$ is primitive if $\Gamma(\mathcal{A})$ is primitive, or equivalently, if the index of imprimitivity of $\mathcal{A}$ is one.

The study of PEP sign patterns was introduced in [1], where it was shown that if $\mathcal{A}^+$ is primitive, then $\mathcal{A}$ is PEP, and where the first example of a PEP sign pattern...
with reducible positive part was given in: the $3 \times 3$ pattern

$$B = \begin{bmatrix} + & - & 0 \\ + & 0 & - \\ - & + & + \end{bmatrix}.$$ 

In Section 2 we extend the results of [1] by generalizing the $3 \times 3$ pattern $B$ from [1] to a family of PEP sign patterns having reducible positive part for every order $n \geq 3$.

In section 3 we examine the effect of the Kronecker product on PEP sign patterns and obtain another method of constructing PEP sign patterns with reducible positive part.

2. A family of sign patterns generalizing $B$. The sign pattern $B$ from [1] was the first PEP sign pattern with a reducible positive part. This sign pattern may be generalized by defining the $n \times n$ sign pattern

$$B_n = \begin{bmatrix} + & - & \cdots & - & 0 \\ + & 0 & \cdots & 0 & - \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ + & 0 & \cdots & 0 & - \\ - & + & \cdots & + & + \end{bmatrix}.$$ 

The following result, which is a special case of the Schur-Cohn Criterion (see, e.g., [5]), will be used in the proof that $B_n$ is PEP.

**Lemma 2.1.** If the polynomial $f(x) = x^2 - \beta x + \alpha$ satisfies $|\beta| < 1 + \alpha < 2$, then all zeros of $f(x)$ lie strictly inside the unit circle.

It is well known that if the characteristic polynomial of $A$ is $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ then $a_{n-k} = (-1)^kE_k(A)$, where $E_k(A)$ is the sum of the $k \times k$ principal minors of $A$ (see, e.g., [3]).

**Theorem 2.2.** For $n \geq 3$ the $n \times n$ sign pattern $B_n$ is PEP.

**Proof.** For $t > 0$, let $B_n(t)$ be the $n \times n$ matrix

$$B_n(t) = \begin{bmatrix} 1 + (n-2)t & -t & \cdots & -t & 0 \\ 1 + t & 0 & \cdots & 0 & -t \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 + t & 0 & \cdots & 0 & -t \\ -(n-2)t - \frac{1}{2}t^2 & t & \cdots & t & 1 + \frac{1}{2}t^2 \end{bmatrix}.$$ 

Then $B_n(t) \in Q(B_n)$, and 1 is an eigenvalue of $B_n(t)$ with positive right eigenvector $1$ (the all ones vector) and positive left eigenvector $w = [(2n-5)/t, 1, \cdots , 1, (2n-4)/t]^T$. 

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We show that for some choice of \( t > 0 \), 1 is a simple strictly dominant eigenvalue of \( B_n(t) \) and hence \( B_n(t) \) is eventually positive. Since \( 1 \in \sigma(B_n(t)) \) and rank \( B_n(t) \leq 3 \), the characteristic polynomial \( p_{B_n(t)}(x) \) of \( B_n(t) \) is of the form

\[
p_{B_n(t)}(x) = x^{n-3}(x-1)(x^2 - \beta x + \alpha) = x^n - (1 + \beta)x^{n-1} + (\alpha + \beta)x^{n-2} - \alpha x^{n-3}.
\]

Computing \( \alpha \) and \( \beta \) using the sums of principal minors to evaluate the characteristic polynomial gives

\[
\beta = \frac{1}{2} t^2 + (n-2) t + 1 \quad \text{and} \quad \alpha = (n-2)t(1 + 2t + \frac{1}{2}t^2). \quad \text{For } n > 3, \text{ setting } t = \frac{1}{2} (n - 2) \text{ gives } |\beta| < 1 + \alpha < 2 \text{ which, using Lemma 2.1, guarantees that the two non-zero eigenvalues of } B_n \text{ other than 1 have modulus strictly less than 1 (recall that a } 3 \times 3 \text{ eventually positive matrix } B_3 \in Q(B_3) \text{ was given in [1] so we have not been concerned with this case in choosing } t).\]

\[\Box\]

We illustrate this theorem with an example.

**Example 2.3.** Let \( n = 5 \). Following the proof of Theorem 2.2, we choose \( t = \frac{1}{6} \) and define

\[
B_5 = B_5(1/6) = \frac{1}{6} \begin{bmatrix} 9 & -1 & -1 & -1 & 0 \\ 7 & 0 & 0 & 0 & -1 \\ 7 & 0 & 0 & 0 & -1 \\ -\frac{37}{12} & 1 & 1 & 1 & \frac{73}{12} \end{bmatrix}.
\]

Moreover, we have

\[
\sigma(B_5) = \left\{ 1, \frac{1}{144} \left( 109 + i \sqrt{2087} \right), \frac{1}{144} \left( 109 - i \sqrt{2087} \right), 0, 0 \right\}
\]

\[
\approx \{ 1, 0.7569 + 0.3172i, 0.7569 - 0.3172i, 0, 0 \},
\]

and \([1 1 1 1]^T \text{ and } \left[ \frac{5}{6} \frac{1}{36} \frac{1}{36} \frac{1}{36} \right]^T \) are right and left eigenvectors corresponding to \( \rho(B_5) = 1 \) respectively. Therefore \( B_5 \) and \( B_5^T \) have the strong Perron-Frobenius property, so \( B_5 \) is eventually positive by Handelman’s criterion.

In [1] it was shown that if the sign pattern \( \mathcal{A} \) is PEP, then any sign pattern achieved by changing one or more zero entries of \( \mathcal{A} \) to be non-zero is also PEP. Applying this to \( B_n \) yields a variety of additional PEP sign patterns having reducible positive part.

3. **Kronecker products.** The Kronecker product (sometimes called the tensor product) is a useful tool for generating larger eventually positive matrices and thus PEP sign patterns. The **Kronecker product** of \( A = [a_{ij}] \) and \( B = [b_{ij}] \) is defined as

\[
A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}.
\]
It is clear that if $A > 0$ and $B > 0$, then $A \otimes B > 0$. The following facts can be found in many linear algebra books, (see, e.g., [6]). For $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times m}$, $(A \otimes B)^k = A^k \otimes B^k$. For $A, C, B,$ and $D$ of appropriate dimensions, $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$. There exists a permutation matrix $P$ such that $B \otimes A = P(A \otimes B)P^T$.

**Proposition 3.1.** If $A$ and $B$ are eventually positive matrices, then $A \otimes B$ is eventually positive.

**Proof.** Assume $A$ and $B$ are eventually positive matrices. Since $A$ and $B$ are eventually positive, there exists some $s_0, t_0 \in \mathbb{Z}$, with $s_0, t_0 > 0$, such that for all $s \geq s_0$ and $t \geq t_0$, $A^s > 0$ and $B^t > 0$. Set $k_0 = \max\{s_0, t_0\}$. Then for all $k \geq k_0$, $(A \otimes B)^k = A^k \otimes B^k > 0$. □

**Corollary 3.2.** If $A$ and $B$ are PEP sign patterns, then $A \otimes B$ is PEP.

If either $A$ or $B$ is a reducible matrix, then $A \otimes B$ is reducible since, without loss of generality, if

$$PAP^T = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

then

$$(P \otimes I)(A \otimes B)(P \otimes I)^T = \begin{bmatrix} A_{11} \otimes B & 0 \\ A_{21} \otimes B & A_{22} \otimes B \end{bmatrix}.$$ 

Thus Corollary 3.2 provides another way to construct PEP sign patterns having reducible positive part.

**Example 3.3.** Let $B = \frac{1}{100} \begin{bmatrix} 130 & -30 & 0 \\ 130 & 0 & -30 \\ -31 & 30 & 101 \end{bmatrix}$. In [1] it was shown that $B$ is eventually positive, and in fact $B^k > 0$ for $k \geq 10$.

Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$. Then $A^k > 0$ for $k \geq 2$, hence $A$ is eventually positive.

Then

$$B \otimes A = \frac{1}{100} \begin{bmatrix} 260 & 390 & -60 & -90 & 0 & 0 \\ 130 & 0 & -30 & 0 & 0 & 0 \\ 260 & 390 & 0 & 0 & -60 & -90 \\ 130 & 0 & 0 & 0 & -30 & 0 \\ -62 & -93 & 60 & 90 & 202 & 303 \\ -31 & 0 & 30 & 0 & 101 & 0 \end{bmatrix}.$$
Moreover \((B \otimes A)^{10} > 0\) and \((B \otimes A)^{11} > 0\), so \(B \otimes A\) is eventually positive and \(\text{sgn}(B \otimes A)\) is a PEP sign pattern with reducible positive part.

Any 0 in \(\text{sgn}(B \otimes A)\) from Example 3.3 may be changed to \(-\) to get yet another PEP sign pattern with reducible positive part.

REFERENCES


